1. Arithmetic Progressions

- **Sequence:** A sequence is an arrangement of numbers in definite order according to some rule.
  Also, we define a sequence as a function whose domain is the set of natural numbers or some subset of the type $\{1, 2, 3\ldots k\}$.
    - A sequence containing finite number of terms is called a finite sequence.
    - sequence containing infinite number of terms is called an infinite sequence.

- A general sequence can be written as

  $a_1, a_2, a_3 \ldots a_{n-1}, a_n, \ldots$

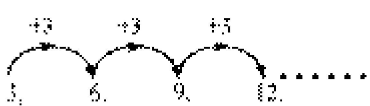
  Here, $a_1, a_2 \ldots$ etc. are called the terms of the sequence and a_n is called the general term or n[th] of the sequence.

- **Fibonacci sequence:** An arrangement of numbers such as 1, 2, 4, 6, 10 … has no visible pattern.
  However, the sequence is generated by the recurrence relation given by
  $a_1 = 1, a_2 = 2, a_3 = 4$
  $a_n = a_{n-2} + a_{n-1}, n > 3$
  This sequence is called the Fibonacci sequence.

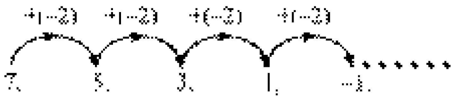
- Let $a_1, a_2, \ldots a_n, \ldots$ be a given sequence. Accordingly, the sum of this sequence is given by the expression
  $a_1 + a_2 + \ldots + a_n + \ldots$
  This is called the series associated with the given sequence.
  The series is finite or infinite according as the given sequence.
  A series is usually represented in a compact form using sigma notation ($\sum$).
  This means the series $a_1 + a_2 + \ldots + a_{n-1} + a_n \ldots$ can be written as $\sum_{k=1}^{n} a_k$.

- **The Concept of Arithmetic Progression**
    - An arithmetic progression is a list of numbers in which the difference between any two consecutive terms is equal.
    - In an AP, each term, except the first term, is obtained by adding a fixed number called common difference to the preceding term.
    - The common difference of an AP can be positive, negative or zero.

  Example 1:

  1. 

  is an AP whose first term and common difference are 3 and 3 respectively.

2. 

is an AP whose first term and common difference are 7 and –2 respectively.

3. 

is an AP whose first term and common difference are –7 and 0 respectively.

- The general form of an AP can be written as a, $a + d$, $a + 2d$, $a + 3d$ …, where a is the first term and d is the common difference.
- A given list of numbers i.e., a_1, a_2, a_3 … forms an AP if $a_{k+1} - a_k$ is the same for all values of k.

Example 2:

Which of the following lists of numbers forms an AP? If it forms an AP, then write its next three terms.

(a) –4, 0, 4, 8, …

(b) 2, 4, 8, 16, …

Solution:

(a) –4, 0, 4, 8, …

$a_2 - a_1 = 0 - (-4) = 4$
$a_3 - a_2 = 4 - 0 = 4$
$a_4 - a_3 = 8 - 4 = 4$
$a_{n+1} - a_n = 4$; for all values of n

Therefore, the given list of numbers forms an AP with 4 being its common difference.
The next three terms of the AP are 8 + 4 = 12, 12 + 4 = 16, 16 + 4 = 20
Hence, AP: –4, 0, 4, 8, 12, 16, 20 …

(b) 2, 4, 8, 16, …

$a_2 - a_1 = 4 - 2 = 2$
$a_3 - a_2 = 8 - 4 = 4$
$a_3 - a_2 \neq a_2 - a_1$

Therefore, the given list of numbers does not form an AP.

- **The terminology related to arithmetic progression**
  - An arithmetic progression is a list of numbers in which each term is obtained by adding a fixed number to the preceding term except the first term.
  - The fixed number is called the common difference (d) of the A.P. The common difference can be either positive or negative or zero.
- **The general form of an A.P.**
  - a, $(a + d)$, $(a + 2d)$, $(a + 3d)$, ... , $[a + (n - 1)d]$, ... where *a is the* first term and *d is* common difference
- **Type of AP**
  - Finite AP: The APs have finite number of terms.
  - Infinite AP: The APs have not finite number of terms.

- In an A.P., except the first term, all the terms can be obtained by adding the common difference to the previous term.
- In an A.P., except the last term, all the terms can be obtained by subtracting the common difference from its subsequent term.

Example:

Find the first four terms of an A.P. whose first term is 9 and the common difference is 6.

Solution:

$a = 9, d = 6$

$a_2 = a + d = 9 + 6 = 15$

$a_3 = a + 2d = 9 + 2 \times 6 = 9 + 12 = 21$

$a_4 = a + 3d = 9 + 3 \times 6 = 9 + 18 = 27$

The first four terms are 9, 15, 21, 27.

- **n^{th} term of an AP**

  The n^{th} term (a_n) of an AP with first term a and common difference d is given by $a_n = a + (n - 1) d$. Here, a_n is called the general term of the AP.

- **n^{th} term from the end of an AP**

  The n^{th} term from the end of an AP with last term l and common difference d is given by $l - (n - 1) d$.

  Example:
  Find the 12^{th} term of the AP 5, 9, 13 …
  Solution:
  Here, $a = 5$, $d = 9 - 5 = 4$, $n = 12$

  $$a_{12} = a + (n - 1) d$$
  $$= 5 + (12 - 1) 4$$
  $$= 5 + 11 \times 4$$
  $$= 5 + 44$$
  $$= 49$$

- **Sum of *n* terms of an AP**
  - The sum of the first n terms of an AP is given by $S_n = \frac{n}{2}[2a + (n - 1)d]$ Sn=n22a+n-1d , where a is the first term and d is the common difference.

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  - If there are only n terms in an AP, then $S_n = \frac{n}{2}[a + l]$ Sn=n2a+l, where $l = a_n$ is the last term.

Example :

Find the value of 2 + 10 + 18 + …. + 802.

Solution:

2, 10, 18… 802 is an AP where $a = 2$, $d = 8$, and $l = 802$.

Let there be n terms in the series. Then,

$a_n = 802$

$\Rightarrow a + (n - 1)\, d = 802$

$\Rightarrow 2 + (n - 1)\, 8 = 802$

$\Rightarrow 8(n - 1) = 800$

$\Rightarrow n - 1 = 100$

$\Rightarrow n = 101$

Thus, required sum = $\frac{n}{2}(a + l) = \frac{101}{2}(2 + 802) = 40602$ n2a+l = 10122+802 = 40602

- **Geometric Progression:** A sequence is said to be a geometric progression (G.P.) if the ratio of any term to its preceding term is the same throughout. This constant factor is called the common ratio and it is denoted by r.

- In standard form, the G.P. is written as a, ar, ar^2 ... where, a is the first term and r is the common ratio.

- **General Term of a G.P.:** The n^{th} term (or general term) of a G.P. is given by $a_n = ar^{n-1}$

Example: Find the number of terms in G.P. 5, 20, 80 ... 5120.

Solution: Let the number of terms be n.

Here $a = 5$, $r = 4$ and $t_n = 5120$

n^{th} term of G.P. = ar^{n-1}

$\therefore 5(4)^{n-1} = 5120$

$\Rightarrow 4^{n-1} = \frac{5120}{5} = 1024$

$\Rightarrow (2)^{2n-2} = (2)^{10}$

$\Rightarrow 2n - 2 = 10$

$\Rightarrow 2n = 12$

$\therefore n = 6$

- **Sum of n Term of a G.P.:** The sum of n terms (S_n) of a G.P. is given by

$$S_n = \begin{cases} \frac{a(1-r^n)}{1-r}, \text{ if } r < 1 \quad \text{or} \quad \frac{a(r^n-1)}{r-1}, \text{ if } r > 1 \\ na, \text{ if } r = 1 \end{cases}$$

Example: Find the sum of the series 1 + 3 + 9 + 27 + ... to 10 terms.

Solution: The sequence 1, 3, 9, 27, ... is a G.P.

Here, $a = 1$, $r = 3$.

Sum of n terms of G.P. = $\frac{a(r^n-1)}{r-1}$ $[r > 1]$

$S_{10} = 1 + 3 + 9 + 27 + \ldots$ to 10 terms

$= \frac{1 \times \left[(3)^{10} - 1\right]}{(3-1)}$

$= \frac{59049-1}{2}$

$= \frac{59048}{2}$

$= 29524$

- Three consecutive terms can be taken as $\frac{a}{r}$, a, ar ar, a, ar. Here, common ratio is r.
- Four consecutive terms can be taken as $\frac{a}{r^3}$, $\frac{a}{r}$, ar, ar^3 ar3, ar, ar, ar3. Here, common ratio is r^2r2.

- **Geometric Mean:** For any two positive numbers a and b, we can insert a number G between them such that a, G, b is a G.P. Such a number i.e., G is called a geometric mean (G.M.) and is given by $G = \sqrt{ab}$

In general, if $G_1, G_2, \ldots, G_n$ be n numbers between positive numbers a and b such that $a, G_1, G_2, \ldots, G_n, b$ is a G.P., then $G_1, G_2, \ldots, G_n$ are given by

$G_1 = ar$, $G_2 = ar^2, \ldots, G_n = ar^n$

Where, r is calculated from the relation $b = ar^{n+1}$, that is $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$.

Example: Insert three geometric means between 2 and 162.
Solution:
Let G_1, G_2, G_3 be 3 G.M.'s between 2 and 162.
Therefor 2, G_1, G_2, G_3, 162 are in G.P.
Let r be the common ratio of G.P.
Here, $a = 2$, $b = 162$ and $n = 3$

$$r=\left(\frac{162}{2}\right)^{\frac{1}{3+1}}=(81)^{\frac{1}{4}}=(3^4)^{\frac{1}{4}}=3$$

$G_1 = ar = 2 \times 3 = 6$
$G_2 = ar^2 = 2 \times (3)^2 = 2 \times 9 = 18$
$G_3 = ar^3 = 2 \times (3)^3 = 2 \times 27 = 54$

Thus, the required three geometric means between 2 and 162 are 6, 18, and 54.

- **Relation between A.M. and G.M.:** Let A and G be the respective A.M. and G.M. of two given positive real numbers a and b. Accordingly, $A=\frac{a+b}{2}$ and $G=\sqrt{ab}$.

Then, we will always have the following relationship between the A.M. and G.M.: $A \geq G$

- **Sum of *n*-terms of some special series:**
  - Sum of first n natural numbers
    $1+2+3+\ldots+n=\frac{n(n+1)}{2}$
  - Sum of squares of the first n natural numbers
    $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$
  - Sum of cubes of the first n natural numbers
    $1^3+2^3+3^3+\ldots+n^3=\left[\frac{n(n+1)}{2}\right]^2$

Example: Find the sum of n terms of the series whose n^{th} term is $n(n+1)(n-2)$.

Solution: It is given that

$$\begin{aligned} a_n &= n(n+1)(n-2) \\ &= n(n^2+n-2n-2) \\ &= n(n^2-n-2) \\ &= n^3-n^2-2n \end{aligned}$$

Thus, the sum of n terms is given by

$$S_n = \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k$$

$$= \left[\frac{n(n+1)}{2}\right]^2 - \frac{n(n+1)(2n+1)}{6} - \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} - \frac{2n+1}{3} - 2\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n(n+1)-2(2n+1)-12}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n-4n-2-12}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2-n-14}{6}\right]$$

$$= \frac{n(n+1)(3n^2-n-14)}{12}$$

$$= \frac{n(n+1)(3n^2-7n+6n-14)}{12}$$

$$= \frac{n(n+1)[n(3n-7)+2(3n-7)]}{12}$$

$$= \frac{n(n+1)(n+2)(3n-7)}{12}$$